Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

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SIMILARITY IN THE ASYMPTOTIC BEHAVIOR OF COLLISION-FREE HYDROMAGNETIC WAVES AND WATER WAVES

by

C. S. Gardner and G. K. Morikawa

May 1, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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ABSTRACT

A similarity is found between the behavior of certain waves in a plasma in a magnetic field and gravity waves on the surface of water of finite depth; in both cases, the particular motion studied in this report is caused by an uniformly moving piston, started impulsively from rest. This similarity in behavior develops asymptotically in time away from the initial state, both in the linearized description of the motion and in a more general non-linear small-perturbation description which includes the linear asymptotic motion. The approximate non-linear theory yields the third order differential equation $2w_\tau + 3ww_\xi + w_{\xi\xi\xi} = 0$ which can be interpreted as describing a reversible dispersion process, in contrast to Burgers' well-known second order differential equation which describes an irreversible diffusion process.

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SIMILARITY IN THE ASYMPTOTIC BEHAVIOR OF COLLISION-FREE HYDROMAGNETIC WAVES AND WATER WAVES

1. INTRODUCTION

In the main body of this report, we study the motion of a collision-free plasma in a magnetic field caused by a piston, started impulsively from rest, moving with a small uniform velocity. In particular we are interested in the asymptotic time-dependent behavior near the equilibrium state. Since the study of the corresponding motion of water waves is so similar, the description is given in the Appendices. An interesting difference, however, is that the original water wave motion is two-dimensional and becomes quasi-one-dimensional asymptotically for large time.

We consider an idealized fluid model* of singly-charged particles moving in an electromagnetic field. The main properties of this model are:

i) Charge neutrality -- the basic property of a plasma,

ii) No collisions -- the charged particles interact only through the self-consistent electromagnetic field, and

iii) Zero-temperature plasma -- completely-ordered motion, in which the fluid motion is the same as the particle motion.

The physical implications of these assumptions and a study, primarily of equilibrium solutions, are described by Adlam and Allen** (1958a), Davis, Lust and Schluter (1958) and others.***

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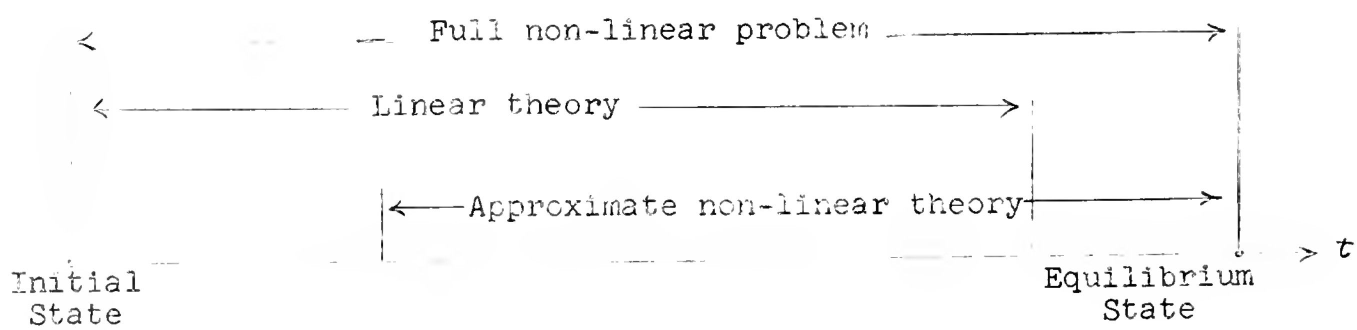
* Some properties and solutions of this model have been summarized by Gardner et al at the Geneva Conference (1958).

** Also some related calculations describing behavior near the initial state of a pinched discharge are given in a contribution to the Geneva Conference (1958b).

*** E.g., A. Banos and M. H. Mittelman at UCRL, Livermore, and M. D. Kruskal at Princeton (private communication).

The study of water waves is an old discipline and the differential equations governing their motion have been known for many years. Actually however, the difficult mathematical analysis of basic non-linear free-boundary type problem is still far from a fully-developed state, e.g. see the treatise by Stoker (1957). The formulation for water of finite depth is summarized in Appendix II where the non-linear approximation is carried out.

The motion which we study of the plasma and water waves may be only roughly similar near the initial state; and the analogy between the two phenomena develops only after some time has passed and the motion is approaching the equilibrium state. This similarity in behavior exists asymptotically in the linearized description (§ 3 and Appendix I), as well as in a time-dependent non-linear approximation (§ 4 and Appendix II) which includes both the linearized asymptotic motion and the final equilibrium state. The range of validity of linearized theory and the approximate non-linear theory is indicated graphically:



2. FORMULATION

As a consequence of the basic property of charge neutrality, the motion of a cold plasma consisting of singly-charged particles can be

described by a one-fluid theory. The state of the plasma is characterized by a system of five first-order partial differential equations for the variables (n, u, v, B, E) where

$n = n_+ = n_-$, number density of ions (or electrons)

$u = u_+ = u_-$, x-component of the ion (or electron) velocity

$v = (v_+ - v_-)$, difference between the y-component of ion and electron velocities

B, z-component of the magnetic field

E, y-component of the electric field

and, with the exception of $E^{(x)}$, x-component of the electric field (charge separation field), all other components are zero. These six quantities depend on the single space coordinate x and the time t. For convenience we consider these quantities to be normalized in the following way:

$$n: \quad n_o$$

$$u: \quad A_o = [B_o^2/\mu n_o(m_+ + m_-)]^{1/2}$$

$$v: \quad a_o = [B_o^2(m_+ + m_-)/\mu n_o m_+ m_-]^{1/2}$$

$$B: \quad B_o$$

$$E: \quad E_o = A_o B_o$$

$$E^{(x)}: \quad E_o^{(x)} = a_o B_o$$

$$x: \quad x_o = [m_+ m_-/\mu e_+{}^2 n_o (m_+ + m_-)]^{1/2}$$

$$t: \quad t_o = [m_+ m_-/e_+{}^2 B_o{}^2]^{1/2} = x_o/A_o$$

where μ is the magnetic permeability of space, e_+ and m_+ are the charge and mass of the ion (minus sign for electrons), and the subscript (zero) refers to a constant initial or boundary state. Then the resulting normalized equations are one conservation equation of number

density and two of momentum and Maxwell's equations (two):

$$n_t + (nu)_x = 0 \tag{1}$$

$$u_t + uu_x = vB \tag{2}$$

$$v_t + uv_x = E - uB \tag{3}$$

$$B_t + E_x = 0 \tag{4}$$

$$B_x = -nv \tag{5}$$

where the displacement current is neglected in (5) and lettered subscripts indicate partial differentiation. In this one-fluid theory, the x-component of the electric field is

$$E^{(x)} = \left(\frac{m_+ - m_-}{m_+ + m_-}\right) vB \tag{6}$$

Also by eliminating the y-component of the electric field E between (3) and (4) we get a simplification. Then

$$(v_x + B)_t + [u(v_x + B)]_x = 0 \tag{7}$$

Comparing (7) with (1), $(v_x + B)$ satisfies the same equation as n. Thus we can set

$$v_x = n - B \tag{8}$$

if this is true initially. Now we have a reduced, but equivalent, system of four equations (1), (2), (5) and (8) for (n, u, v, B) in place of the original five and (3) may be regarded as the defining equation for E. We note that the particle path and the plane t = constant are the only set of real characteristics (actually, a double set) for this differential equation system; and neither A_o nor a_o are characteristic velocities in this sense.

For later use (§ 4) we look for equilibrium solutions of the reduced system (1), (2), (5) and (8), noting that the equations are Gallilean invariant, i.e., invariant under the transformation: $\{x \longrightarrow (x - u_o t),\ t \longrightarrow t,\ u \longrightarrow (u - u_o)\}$ where u_o is constant. Setting $\frac{\partial}{\partial t} = 0$ in (1) and (2), combined with (5) and (8), three integrals of mass, momentum and energy follow by inspection

$$nu = u_o \tag{9}$$

$$u_o u + \frac{B^2}{2} = u_o^2 + \frac{1}{2} \tag{10}$$

$$\frac{1}{2}(u^2 + v^2) + B = \frac{u_o^2}{2} + 1 \tag{11}$$

Combining (10) and (11) yields the "phase relation"

$$v^2 = (B-1)^2 [1 - \frac{(B+1)^2}{4u_o^2}] \tag{12}$$

Eliminating n, u, and v by (5), (9), (10) and (12), $B(x)$ can be integrated explicitly (cf. references in § 1). We obtain the solution in the limiting case of a small disturbance by the transformation

$\xi = \varepsilon^{1/2}x$ and letting $n = 1 + \varepsilon n^{(1)}(\xi)$, $u = 1 + \varepsilon u^{(1)}(\xi)$, $v = \varepsilon^{3/2}v^{(1)}(\xi)$, $B = 1 + \varepsilon B^{(1)}(\xi)$ and $u_o = 1 + \varepsilon a$, where a = const. to be determined and $\varepsilon = (B_{max}-1) \ll 1$ is the magnetic field amplitude. Then (5), (9), (10), and (12) become

$$\frac{dB^{(1)}}{d\xi} = -v^{(1)} \tag{13}$$

$$n^{(1)} + u^{(1)} = a \tag{14}$$

$$u^{(1)} + B^{(1)} = a \tag{15}$$

$$(v^{(1)})^2 = (B^{(1)})^2(2a - B^{(1)}) \tag{16}$$

Eliminating $v^{(1)}$ between (13) and (16), $B^{(1)}$ satisfies

$$\frac{dB^{(1)}}{d\xi} = -B^{(1)}(2a - B^{(1)})^{1/2} \tag{17}$$

which yields the equilibrium solution

$$B^{(1)} = 2a \operatorname{sech}^2(a\xi) \tag{18}$$

We choose $a = 1/2$ so that $B^{(1)}_{max} = 1$, consistent with our definition of ε. (18) is a pulse-type solution similar to the small-amplitude solitary water wave, e.g. see Keller (1948). The boundary conditions chosen in (9), (10) and (11) have eliminated some additional equilibrium solutions (periodic) which can be recovered from the time-dependent non-linear approximation derived in § 4.

3. LINEAR APPROXIMATION*

To describe the motion caused by small initial disturbances we linearize the reduced plasma equations (1), (2), (5) and (8) by perturbing on the rest state $(n,u,v,B) = (1,0,0,1)$. The resulting equations in the first approximation are

$$n_t^{(1)} + u_x^{(1)} = 0 \tag{19}$$

$$u_t^{(1)} = v^{(1)} \tag{20}$$

$$B_x^{(1)} = -v^{(1)} \tag{21}$$

$$v_x^{(1)} = n^{(1)} - B^{(1)} \tag{22}$$

By elimination we obtain a single fourth order differential equation for the x-component of the velocity $u^{(1)}$:

$$u_{tt}^{(1)} - u_{xx}^{(1)} - u_{xxtt}^{(1)} = 0 \tag{23}$$

Also $n^{(1)}$, $B^{(1)}$ and $v^{(1)}$ satisfy the same equation (23). We now consider the plasma motion for $x > 0$ caused by an electrically-neutral piston** (nominally at $x = 0$) started impulsively from rest

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* The results in this section were presented by Gardner (1958) at the Controlled Thermonuclear Conference in Wash. D. C. and have been summarized in the Proceedings of the Geneva Conference by Gardner et al (1958).

** This idealized problem is considered for convenience. We could just as easily take the same differential equation (23) satisfied by $B^{(1)}$ and consider a magnetically-driven piston.

and moving with uniform velocity U_1 in the positive x direction. Then the initial conditions are

$$u^{(1)}(x,0) = u_t^{(1)}(x,0) = 0 \tag{24}$$

and the boundary conditions for $t > 0$ is

$$\frac{u^{(1)}(0,t)}{U_1} = H(t) \tag{25}$$

where $H(t)$ is the unit step function (= 0 for $x < 0$ and = 1 for $x \geq 0$). Equation (23) with the conditions (24) and (25) can be solved by applying the Laplace transform with respect to the time t. The solution is

$$\frac{u^{(1)}(x,t)}{U_1} = \frac{1}{2\pi i} \int_{\Gamma} \frac{ds}{s} \exp \left\{ts[1-\alpha(1+s^2)^{-1/2}]\right\} \tag{26}$$

where $\alpha = x/t$ and the contour Γ goes from $-i\infty$ to $+i\infty$ in the right half of the complex s-plane. To carry out the inverse Laplace transform (26) appears difficult; but the asymptotic behavior for large t can be determined by the method of steepest descent. Then the asymptotic form of the velocity distribution for large t (and x) is

$$\frac{u^{(1)}(x,t)}{U_1} \backsim \int_{\beta}^{\infty} A_i(\alpha)\, d\alpha \tag{27}$$

where $\beta = (2/3)^{1/3}(x-t)/t^{1/3}$ and

$$A_i(\beta) = \frac{1}{\pi} \int_0^\infty d\lambda \cos\left(\frac{\lambda^3}{3} + \beta\lambda\right) \tag{28}$$

is the Airy function. The same asymptotic behavior is found for the linearized motion of water waves in Appendix I, equation 1-17. Equation (27) describes a shock-type velocity distribution with damped oscillations behind the "shock" front ($\beta < 0$). The velocity decreases exponentially ahead of the front $\beta \geq 0$, and for negative β the velocity oscillates about and approaches the piston velocity U_1 as $\beta \longrightarrow -\infty$. Also the amplitude of the oscillation decreases like $\beta^{-3/4}$ as $\beta \rightarrow -\infty$. However this wave is not a shock in the usual sense since it is non-stationary, widening with time like $t^{1/3}$; thus (27) describes a slowly dispersing wave.

4. NON-LINEAR APPROXIMATION

So far the only apparent relationship between the linearized asymptotic behavior of plasma motion (equation (27) in § 3) and the pulse-like equilibrium solution (equation (18) in o 2) is their propagation speed $A_o = [B_o{}^2/\mu n_o(m_+ + m_-)]^{1/2}$ as $\varepsilon \longrightarrow 0$. In this section we seek that approximate time-dependent description which is valid for large time, i.e., valid under the conditions that it 1) includes the equilibrium solution (18) and 2) is in the same asymptotic direction in the x-t plane as the linearized asymptotic solution (27), that is, the direction defined by $\beta = (x-t)/t^{1/3} =$ const. It is clear from (17) and (18) that a non-linear approximation is required. A comparable formulation has been found in the approximate description of the motion of flood waves in rivers by Morikawa (1957).

To obtain the desired non-linear approximation of the plasma equations (1), (2), (5) and (8), we first apply the above condition 1) which states that the pulse solution (18) is to be included in our approximate formulation. Thus, we make the following scale transformations:

$$\xi = \varepsilon^{1/2}(x-t) \tag{29}$$

and

$$v = \varepsilon^{1/2}V. \tag{30}$$

Condition 2) above fixes the necessary scale transformation of time:

$$\tau = \varepsilon^{3/2}\ t \tag{31}$$

since we want $(x-t)/t^{1/3}$ to be invariant with respect to the coordinate transformations (29) and (31). In addition, we make a perturbation expansion with respect to ε on the rest state $(n,u,v,B) = (1,0,0,1)$:

$$n = 1 + \varepsilon n^{(1)}(\xi,\tau) + \varepsilon^2 n^{(2)}(\xi,\tau) + \dots \tag{32}$$

$$u = \varepsilon u^{(1)}(\xi,\tau) + \varepsilon^2 u^{(2)}(\xi,\tau) + \dots \tag{33}$$

$$V = \varepsilon V^{(1)}(\xi,\tau) + \varepsilon^2 V^{(2)}(\xi,\tau) + \dots \tag{34}$$

$$B = 1 + \varepsilon B^{(1)}(\xi,\tau) + \varepsilon^2 B^{(2)}(\xi,\tau) + \dots \tag{35}$$

By the scale transformations (29), (30) and (31), the equations (1), (2), (5) and (8) become

$$\varepsilon n_\tau + [n(u-1)]_\xi = 0 \tag{36}$$

$$\varepsilon u_\tau + (u-1)u_\xi = 0 \tag{37}$$

$$B_\xi = -nV \tag{38}$$

$$\varepsilon V_\xi = n-B \tag{39}$$

Putting (32 to 35) in (36 to 39) and equating the coefficients of powers of ε in the usual way, we obtain the equations which the lowest order approximation $(n^{(1)}, u^{(1)}, V^{(1)}, B^{(1)})$ satisfies:

$$n^{(1)} - u^{(1)} = 0 \tag{40}$$

$$u_\xi^{(1)} = B_\xi^{(1)} = -V^{(1)} \tag{41}$$

$$n^{(1)} - B^{(1)} = 0 \tag{42}$$

and

$$(n^{(2)} - u^{(2)})_\xi = n_\tau^{(1)} + (n^{(1)}u^{(1)})_\xi \tag{43}$$

$$u_\xi^{(2)} + V^{(2)} = u_\tau^{(1)} + u^{(1)}u_\xi^{(1)} - V^{(1)}B^{(1)} \tag{44}$$

$$B_\xi^{(2)} + V^{(2)} = -n^{(1)}V^{(1)} \tag{45}$$

$$n^{(2)} - B^{(2)} = V_\xi^{(1)} \tag{46}$$

The second order approximation terms $(n^{(2)}, u^{(2)}, V^{(2)}, B^{(2)})$ may be eliminated among (43 to 46) and $u^{(1)}$ (or $n^{(1)}$ or $B^{(1)}$) satisfies the third order non-linear equation

$$2u_{\tau}^{(1)} + 3u^{(1)}u_{\xi}^{(1)} + u_{\xi\xi\xi}^{(1)} = 0 \tag{47}$$

(47) describes the plasma motion traveling at the velocity $A_o = [B_o^2/\mu n_o(m_+ + m_-)]^{1/2}$ in the positive x direction near and including the equilibrium state. An equation of the same form describes the motion of water waves traveling at the speed of small gravity waves (Appendix II, equation (II-32)).

In contrast to Burgers' second order equation* which describes an irreversible diffusion process, (47) describes a dispersion process which is reversible in the following sense: the equation is invariant with respect to change of sign $\xi \longrightarrow -\xi$ and $\tau \longrightarrow -\tau$. The solution of the initial value problem with a step-function initial distribution $u^{(1)}(\xi,0) = \frac{1}{2} - H(\xi)$ ($H = 0$ for $\xi < 0$ and $H = 1$ for $\xi \geq 0$) would be of some interest. But, the Hopf (1950)- Cole (1951) technique, which yields the corresponding solution of Burgers' equation, apparently does not apply here. We plan to carry out a numerical integration.

It is easily shown how the linearized asymptotic solution (27) is related to (47) and that the pulse solution (18) is included in the equilibrium solutions of (47). Neglecting the non-linear term in (47) we look for similarity solutions of

$$2u_{\tau}^{(1)} + u_{\xi\xi\xi}^{(1)} = 0 \tag{48}$$

of the form $u^{(1)} = \tau^m f(\beta)$ where $\beta = b\xi\tau^n$ and b, m and n are constants. Then, m is arbitrary and $n = -1/3$ and f satisfies the ordinary differential equation

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* Burgers' equation is $w_t + w\, w_x = \nu\, w_{xx}$ where ν = const.

$$f''' - \beta f' + 3mf = 0 \tag{49}$$

where $b = (2/3)^{1/3}$ for convenience. We consider three particular values of $m = 0, -1/3, -2/3$:

Case 1, $m = 0$. For this case (49) becomes

$$(f')'' - \beta(f') = 0 \tag{50}$$

and f' satisfies the Airy equation so that

$$f' = A_i(\beta) \tag{51}$$

The linearized asymptotic solution (27) corresponds to this case.

Case 2, $m = -1/3$. For this case a solution of (49) is

$$f = A_i(\beta) \tag{52}$$

and

$$u^{(1)} = \tau^{-1/3} \, A_i(\beta) \tag{53}$$

(53) is the asymptotic solution obtained by Jeffreys and Jeffreys (1956) in their study of the motion of water waves.

Case 3, $m = -2/3$. For this case a solution of (49) is

$$f = A_i'(\beta) \tag{54}$$

(53) is unbounded for $\beta \to -\infty$ but this case is of interest since the non-linear equation (47) also has a similarity solution of the form $u^{(1)} = \tau^{-2/3} f(\beta)$ where $f(\beta)$ satisfies the equation

$$f''' + ff' - \beta f' - 2f = 0 \tag{55}$$

It would be of interest to determine whether there is a bounded solution of (55) over the entire range of $-\infty \le \beta \le \infty$. Equilibrium solutions of (47) are obtained by seeking solutions of the form $u^{(1)} = F(\xi - a\tau)$ where a = const. to be determined. Since $(\xi - a\tau) = \varepsilon^{1/2}[x - (1+\varepsilon a)t]$, (εa) is the velocity deviation of the steady progressing wave from the critical speed A_o. By (47), $F(\xi - a\tau)$ satisfies the equation

$$F''' + 3FF' - 2aF = 0 \tag{56}$$

Multiplying (56) by F' and integrating twice we obtain

$$(F')^2 + F^3 - 2aF^2 = c_1 F + c_2 \tag{57}$$

where c_1 and c_2 are integration constants. If we impose the condition that the solution vanishes at infinity, then $F = F' = F'' = 0$ implies that $c_1 = c_2 = 0$; and (57) becomes

$$(F')^2 = F^2(2a - F) \tag{58}$$

which corresponds to the solitary wave equation (17). If we define ε as the wave amplitude, then $u^{(1)}_{max} = 1$ implies that $a = 1/2$. The flow velocity* is supercritical at infinity and subcritical at the peak. Periodic solutions are obtained by relaxing the conditions at infinity. For example $F = F' = 0$ implies that $c_2 = 0$ and we obtain for fixed amplitude a one-parameter (wave length) family of periodic solutions (cnoidal waves) with speeds less than or equal to the solitary wave speed.

Finally, we summarize the correspondence between variables describing the asymptotic motion of plasma waves and water waves.

Plasma		Water
$n^{(1)}, B^{(1)}$	$n^{(1)}, B^{(1)}$	$h^{(1)}$
$u^{(1)}$		$u^{(1)}$
$v^{(1)}$		$v^{(1)}$
$A_o=[B_o^2/\mu n_o(m_+ + m_-)]^{1/2}$		$(gh_o)^{1/2}$
$x_o=[m_+ m_-/\mu e_+^2 n_o(m_+ + m_-)]^{1/2}$		h_o
$t_o = x_o/A_o$		$(h_o/g)^{1/2}$

We point out that for plasma motion the y-component of velocity $v^{(1)}$ is normalized with respect to a different velocity $a_o = [B_o^2(m_+ + m_-)/\mu n_o m_+ m_-]^{1/2}$ than $u^{(1)}$ which is normalized with respect to A_o (cf. § 2). Thus the actual y-component of velocity exceeds the x-component of velocity by a factor $a_o/A_o = (m_+/m_-)^{1/2}$. $[1 + (m_-/m_+)]$ for $(m_+/m_-) >> 1$.

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* Note that the flow velocity is from left to right with respect to the solitary wave in § 2 and in the opposite direction in this section, appropriate for a piston moving from left to right.

APPENDIX I. LINEAR APPROXIMATION OF MOTION OF WATER WAVES

To compare with the plasma motion described in § 2, we consider the motion caused by a piston (started impulsively) moving at a small uniform speed U_1 into water initially at rest of uniform depth h_o. The piston (nominally at $x = 0$) moves in the positive x direction and the gravitational acceleration g is in the negative y direction. Neglecting viscosity, the motion is described by the velocity potential $\phi(x,y,t)$. For convenience we introduce the following normalization:

$$x: \quad h_o$$

$$t: \quad (h_o/g)^{1/2}$$

$$\phi: \quad (gh_o^3)^{1/2}$$

$$U_1: \quad (gh_o)^{1/2}$$

Since the boundary conditions on the water surface (cf. Appendix II) are linearized, the flow is nominally confined to move in a semi-infinite strip $x \geq 0$, with the horizontal bottom at $y = 0$ and the free surface at $y = 1$. The motion is governed by the Laplace equation in the interior:

$$\phi_{xx} + \phi_{yy} = 0 \tag{I-1}$$

The initial conditions are

$$\phi(x,y,0) = \phi_t(x,y,0) = 0. \tag{I-2}$$

The boundary conditions are: at $x = 0$

$$\frac{\phi_x(0,y,t)}{U_1} = H(t) \tag{I-3}$$

where H(t) is the unit step function; on the bottom y = 0

$$\phi_y(x,0,t) = 0; \tag{I-4}$$

and at the linearized free surface y = 1

$$\phi_{tt} + \phi_y = 0 \tag{I-5}$$

The solution of (I-1), with the conditions (I-2 to 5), is readily obtained by applying the Laplace transform with respect to t and the Fourier cosine transform with respect to x; the resulting second order ordinary differential equation in the remaining variable y is easily solved. After a simple inversion of the Laplace transform, the inverse Fourier transform yields the integral solution

$$\phi(x,y,t) = -\frac{2U_1}{\pi}\int_0^\infty \frac{d\kappa}{\kappa^2}\cos(\kappa x)\left[1-\frac{\cosh(\kappa y)}{\cosh\kappa}\cos[(\kappa\tanh\kappa)^{1/2}t]\right] \tag{I-6}$$

The wave profile $\eta(x,t) = \phi_t(x,1,t)$ and from (I-6)

$$\eta(x,t) = -\frac{2U_1}{\pi}\int_0^\infty \frac{d\kappa}{\kappa^2}(\kappa\tanh\kappa)^{1/2}\cos(\kappa x)\sin[(\kappa\tanh\kappa)^{1/2}t] \tag{I-7}$$

We make an asymptotic evaluation of (I-7) for large t (and x) by a variant of the method of steepest descent. Since the integrand of (I-7) is regular at $\kappa = 0$ and is an even function of κ, then $\int_0^\infty = \frac{1}{2}\int_{-\infty}^\infty$. We write (I-7) as the sum of two sine integrals

$$\cos a \sin b = -\tfrac{1}{2}[\sin(a-b) - \sin(a+b)] \tag{I-8}$$

where $a = \kappa x$, $b = (\kappa\tanh\kappa)^{1/2}t$ and

$$(a \pm b) = \chi t[a \pm (\tanh\chi)^{1/2}/\chi^{1/2}] \tag{I-9}$$

where $a = x/t$. For $\chi << 1$ (large x), (I-9) is approximately

$$(a \pm b) \approx \chi t[a \pm (1 - \frac{\chi^2}{6})] \tag{I-10}$$

For large t and $a = O(1)$ $(x > 0)$, the integral containing $\sin(a-b)$ dominates that with $\sin(a+b)$. By transforming $\chi = (2/t)^{1/3}\lambda$ and $(a-1) = \bar{\beta}/(2t^2)^{1/3}$, we rewrite $(a-b)$ in (I-10)

$$(a-b) \approx \lambda(\frac{\lambda^2}{3} + \bar{\beta})$$

and the asymptotic solution of (I-7) for large x and t is

$$\eta(x,t) \sim \frac{U_1}{2\pi}\int_{C}^{\infty} \frac{d\lambda}{\lambda} \sin(\frac{\lambda^3}{3} + \bar{\beta}\lambda) \tag{I-11}$$

Since the derivative with respect to $\bar{\beta}$ of the integral (I-11) is the Airy function

$$Ai(\bar{\beta}) = \frac{1}{\pi}\int_0^{\infty} d\lambda \cos(\frac{\lambda^3}{3} + \bar{\beta}\lambda) \tag{I-12}$$

we rewrite (I-11) in the concise form

$$\eta(x,t) \sim U_1 \int_{\bar{\beta}}^{\infty} Ai(\gamma)d\gamma \tag{I-13}$$

where

$$\bar{\beta} = 2^{1/3}(x-t)/t^{1/3} \tag{I-14}$$

The wave profile (I-13) is of the same form as (27) in § 2. Comparing (I-14) with (28), $\bar{\beta}$ differs from β by a factor $3^{-1/3}$ (cf Appendix II, equation (II-32)). A more direct comparison with (27) is found by calculating the x-component of velocity $u(x,1,t) = \phi_x(x,1,t)$ at the free surface $y = 1$. From (I-6)

$$\frac{u(x,1,t)}{U_1} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\chi}{\chi} \sin(\chi x) \left\{1-\cos[(\chi \tanh \chi)^{1/2} t]\right\} \tag{I-15}$$

Since

$$\int_{-\infty}^{\infty} \frac{d\chi}{\chi} \sin(\chi x) = \pi, \ x > 0,$$

(I-15) becomes

$$\frac{u(x,1,t)}{U_1} = 1 - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\chi}{\chi} \sin(\chi x) \cos[\chi(\tanh \chi)^{1/2} t] \tag{I-16}$$

which differs from the wave profile (I-7); but the asymptotic form of (I-16) is the same as (I-13) and

$$\frac{u(x,1,t)}{U_1} \sim \int_{\bar{\beta}}^{\infty} Ai(\gamma)d\gamma \tag{I-17}$$

APPENDIX II. ASYMPTOTIC NON-LINEAR APPROXIMATION FOR MOTION OF WATER WAVES

In two dimensions the system of equations governing incompressible flow are the conservation equations of mass and mementum:

$$u_x + v_y = 0 \tag{II-1}$$

$$\frac{du}{dt} + p_x = 0 \tag{II-2}$$

$$\frac{dv}{dt} + p_y + 1 = 0 \tag{II-3}$$

where

$$\frac{d(\,)}{dt} = (\,)_t + u\cdot(\,)_x + v\cdot(\,)_y$$

The variables have been normalized as follows:

x, y: h_o

t: $(h_o/g)^{1/2}$

p: $\rho g h_o$

where h_o is the depth for the rest state, g is the gravitational acceleration and ρ is the density (constant). In addition to (II-1,2,3) we impose the irrotationality condition

$$v_x - u_y = 0 \tag{II-4}$$

The boundary condition at the horizontal bottom is

$$v(x,0,t) = 0; \tag{II-5}$$

and at the free surface

$$v(x,h,t) = \left\{\frac{\partial}{\partial t} + u(x,h,t)\,\frac{\partial}{\partial x}\right\}\,h(x,t) \tag{II-6}$$

and

$$p(x,h,t) = 0. \tag{II-7}$$

Just as for the plasma equations (1), (2), (5) and (8), the particle path and the plane t = constant are each double characteristics of (II-1, 2, 3).

Following the same procedure by which we derived the non-linear approximation of the plasma equations in § 4, we obtain the asymptotic approximation for water waves by 1) including the small amplitude solitary wave as an equilibrium solution and 2) scaling the time so that $(x-t)/t^{1/3}$ is invariant, as dictated by the linearized asymptotic solution (I-17). Then the required scale transformations are the same as those for the plasma equations (cf. (29), (30) and (31))

$$\xi = \varepsilon^{1/2}(x-t) \tag{II-8}$$

$$\tau = \varepsilon^{3/2}t \tag{II-9}$$

$$v = \varepsilon^{1/2}V \tag{II-10}$$

and the remaining variables y, u and p are unchanged. The flow equations (II-1,2,3) are transformed by (II-8,9,10) into

$$u_\xi + V_y = 0 \tag{II-11}$$

$$\frac{du}{d\tau} + p_\xi = 0 \tag{II-12}$$

$$\varepsilon\,\frac{dV}{d\tau} + p_y + 1 = 0 \tag{II-13}$$

where the notation $d(\ )/d\tau$ means

$$\frac{d(\ )}{d\tau} = \varepsilon(\ )_{\tau} + (u-1)(\ )_{\xi} + V\cdot(\ )_{y} \tag{II-14}$$

and the irrotationality condition (II-4) becomes

$$\varepsilon V_{\xi} - u_{y} = 0. \tag{II-15}$$

In addition, the boundary conditions (II-5,6,7) become

$$V(\xi,0,\tau) = 0 \tag{II-16}$$

$$V(\xi,h,\tau) = \left\{\varepsilon \frac{\partial}{\partial\tau} + [u(\xi,h,\tau) - 1] \frac{\partial}{\partial\xi}\right\} h(\xi,\tau) \tag{II-17}$$

and

$$p(\xi,h,\tau) = 0 \tag{II-18}$$

There is a close relationship between the scale transformations (II-8,9,10) and the transformation of variables yielding the well-known long wave (or shallow water) approximation [cf. Friedrichs (1948) or Morikawa (1956)]. Both transformations yield the hydrostatic approximation for $\varepsilon \to 0$, i.e. in (II-13) $\varepsilon \frac{dV}{d\tau} << (p_y + 1)$. Also in Morikawa's version (differs from Friedrichs' in the scaling of the vertical velocity component) the irrotionality condition (II-15) states that $u_y = 0$ for $\varepsilon = 0$, i.e. the horizontal velocity component $u(\xi,\tau)$ is independent of the vertical coordinate y. The scale transformation yielding the long wave approximation differs only in the time scaling, being $\tau = \varepsilon^{1/2}t$ instead of (II-9). That is, the non-linear approximation, which we describe in this Appendix ($\tau = \varepsilon^{3/2}t$), describes the

motion at later time and closer to the final equilibrium state than the long wave approximation.

In addition to the scale transformation we make a perturbation expansion (cf. § 4) on the rest state $(h, u, V, p) = (1, 0, 0, (1-y))$. Then,

$$h = 1 + \varepsilon h^{(1)}(\xi, \tau) + \varepsilon^2 h^{(2)}(\xi, \tau) + \ldots \tag{II-19}$$

$$u = \varepsilon u^{(1)}(\xi, y, \tau) + \varepsilon^2 u^{(2)}(\xi, y, \tau) + \ldots \tag{II-20}$$

$$V = \varepsilon V^{(1)}(\xi, y, \tau) + \varepsilon^2 V^{(2)}(\xi, y, \tau) + \ldots \tag{II-21}$$

$$p = (1-y) + \varepsilon[p^{(1)}(\xi, y, \tau) + h^{(1)}(\xi, \tau)] + \varepsilon^2[p^{(2)}(\xi, y, \tau) + h^{(2)}(\xi, \tau)] + \ldots \tag{II-22}$$

noting that at the free surface $y = h$

$$V(\xi, h, \tau) = \varepsilon V^{(1)}(\xi, 1, \tau) + \varepsilon^2[V^{(2)}(\xi, 1, \tau) + V_y^{(1)}(\xi, 1, \tau) \cdot h^{(1)}(\xi, \tau)] + \ldots \tag{II-23}$$

$$p(\xi, h, \tau) = \varepsilon p^{(1)}(\xi, 1, \tau) + \varepsilon^2[p^{(2)}(\xi, 1, \tau) + p_y^{(1)}(\xi, 1, \tau) \cdot h^{(1)}(\xi, \tau)] + \ldots \tag{II-24}$$

Putting (II-19 to 22) in (II-11 to 18) and equating coefficients of like powers of ε, the lowest and, if desired, higher order approximations can be calculated in a systematic way. We summarize the equations yielding the lowest order approximation $(h^{(1)}, u^{(1)}, V^{(1)}, p^{(1)})$. Then, $u^{(1)}(\xi, \tau)$ is independent of y, $p^{(1)} \equiv 0$ and

$$h^{(1)} - u^{(1)} = 0 \tag{II-25}$$

$$v^{(1)} = -y\, u_{\xi}^{(1)} \; . \tag{II-26}$$

Also,

$$u^{(2)} = -(y^2/2)u^{(1)}_{\xi\xi} + F(\xi,\tau) \qquad \text{(II-27)}$$

$$v^{(2)} = (y^3/6)u^{(1)}_{\xi\xi\xi} - yF_\xi \qquad \text{(II-28)}$$

$$p^{(2)} = \tfrac{1}{2}(1-y^2)u^{(1)}_{\xi\xi} \qquad \text{(II-29)}$$

$$h^{(2)}_\xi = -[(u^{(1)}_{\xi\xi\xi}/6) - F_\xi] + h^{(1)}_\tau + (h^{(1)}u^{(1)})_\xi \qquad \text{(II-30)}$$

$$(p^{(2)} + h^{(2)} - u^{(2)})_\xi = -(u^{(1)}_\tau + u^{(1)}u^{(1)}_\xi) \qquad \text{(II-31)}$$

The second order terms ($u^{(2)}$, $v^{(2)}$, $p^{(2)}$, $h^{(2)}$) and the integration variable (with respect to y) $F(\xi,\tau)$ can be eliminated among (II-27 to 31) resulting in a single equation for $u^{(1)}$:

$$2u^{(1)}_\tau + 3u^{(1)}u^{(1)}_\xi + \frac{1}{3}u^{(1)}_{\xi\xi\xi} = 0 \qquad \text{(II-32)}$$

Except for the factor (1/3) multiplying the third derivative term and which can be eliminated by renormalizing the space and time variables, (II-32) is the same equation as (47) which describes the asymptotic plasma motion. Aside from minor differences, (II-32) was derived many years ago by Korteweg and de Vries*(1895) following an approximation procedure of Rayleigh; our procedure is more direct and indicates more clearly the asymptotic nature of (II-32). Like (47) for plasma waves, the third order non-linear partial differential equation (II-32) describes the motion of water waves traveling with velocity $(gh_o)^{1/2}$

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* Keller's (1948) results are also a rederivation of the small-amplitude equilibrium solutions by scale transformation with respect to a small parameter.

in the positive x direction near and including the equilibrium state. A related fourth order differential equation which approximately describes waves traveling in both positive and negative x directions due to Boussinesq has been rederived recently by Ursell (1953). (II-32) can be derived directly from this fourth order equation by applying the scale transformations (II-8,9) and letting $\varepsilon \longrightarrow 0$.

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